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A CENTRAL LIMIT THEOREM FOR MIXING STATIONARY POINT PROCESSES

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Suppose A_0 is a strictly stationary, second order point process on Z^d that is ϕ-mixing. The particles initially present are then continually subjected to random translations via random walks. If A_n is the point process resulting at time n, then we prove, under certain technical conditions, that the total occupation time by time n of a finite nonempty subset B of Z^d, namely, $S_n(B)=\sum_{k=1}^{n} A_k(B)$, is asymptotically normally distributed.

Central limit theorem
mixing processes
infinite particle systems.
random translations
random walks

1. Introduction

We consider a point process (random counting measure) A_0 on a countable set, that we take to be $\mathbf{Z}^d$. The process evolves through continual independent random motions of the particles (points) initially present.

For B a finite nonempty subset of $\mathbf{Z}^d$ we denote by $S_n(B)$ the total occupation time of B by all particles by time n. Basically, the problem we consider here is to determine a central limit theorem for the functional $S_n(B)$.

In [4], Port proved a central limit theorem for $S_n(B)$ under the assumption that the particles move according to the transition function P of a transient Markov chain with invariant measure μ, and that A_0 is a Poisson process with intensity μ.

Weiss, in [7], further investigated the system, where he established a central limit theorem for $S_n(B)$, with the Poisson assumption no longer in force, but with the independence of the $A_0(x)$, $x \in Z^d$ still assumed. Although the results in [7] are similar to those in [4], the relaxing of the Poisson assumption necessitated the employment of entirely different techniques in establishing the CLT. However, these techniques rested heavily on the independence assumption. It is also interesting to note that, in certain cases, the asymptotic variance of $S_n(B)$ differed in the Poisson and non-Poisson situations.

The purpose of this paper is to establish a central limit theorem for $S_n(B)$ without the assumption of independence on the initial point process. As will be seen, the relaxing of the independence assumption, renders inappropriate the techniques used in both [4] and [7]. Moreover, in dimension $d=1$, the asymptotic variance of $S_n(B)$ differs from those in both [4] and [7].

Specifically, we shall make the following assumptions. We assume that the point process A_0 is a strictly stationary, second order process that is ϕ-mixing. Mixing processes on Z^d are discussed in Section 3. Further we assume that the random motions are obtained through transient strongly aperiodic random walks. The CLT is first established under the assumption of boundedness on A_0, but later it is shown how this restriction can be removed.

2. Preliminaries and notation

Let $\{\xi_n;\ n \geq 0\}$ be independent and, for $n \geq 1$, identically distributed $\mathbf{Z}^d$-valued random variables. Let $Y_n = \sum_{k=0}^{n} \xi_k$, $n \geq 0$, be the associated d-dimensional random walk. We denote by $P_n(x, y)$ the n-step transition function of the random walk. Let $\theta = \{x: P_1(0, x) > 0\}$. The random walk is called *strongly aperiodic* if for each x, the group generated by $x + \theta$ is $\mathbf{Z}^d$. We also shall employ the following notation.

$$P_n(x, B) = \sum_{y \in B} P_n(x, y),$$

$$G_n(x, B) = \sum_{k=1}^{n} P_k(x, B),$$

$$G(x, B) = \sum_{k=1}^{\infty} P_k(x, B).$$

If A is a second order, strictly stationary process on Z^d, we shall denote its covariance function by C. That is, $C(x) = \operatorname{cov}(A(y), A(x+y))$ for $x, y \in \mathbf{Z}^d$.

Now, if A_0 is the initial point process, we let A_n denote the point process resulting at time n from the random motions of the particles. As special cases of Theorems 1 and 4 of [3], we have the following result that will be used in the sequel.

Theorem 2.1. *If A_0 is covariance stationary, then so is A_n, $n \geq 1$. Moreover, $\mathbf{E}A_n(x)$ is independent of n (and, of course, of x).*

3. Mixing processes

For $S, T \subset \mathbf{Z}^d$, define $\rho(S, T) = \inf\{\|x-y\|_\infty: x \in S, y \in T\}$. Now, let B be a strictly stationary process on $\mathbf{Z}^d$. For $S, T \subset \mathbf{Z}^d$, let $\mathcal{S}(\mathcal{T})$ be the σ-field generated by the r.v.'s

$B(x)$, $x \in S(T)$. Let ϕ be a nonnegative function defined on the positive integers. We say that the process B is *ϕ-mixing* if

$$\sup_{A\in\mathcal{S},B\in\mathcal{T}} |P(B|A)-P(B)|| \leq \phi(k) \tag{3.1}$$

whenever $\rho(S, T)=k$ $(k \geq 1)$.

It will be convenient to define $\phi(0)=1$, and we do so. Note that if we replace $\phi(k)$ by $\psi(k)=\min\{1, \phi(1), \ldots, \phi(k)\}$, then B is ψ-mixing. Therefore, without loss of generality we assume that ϕ is nonincreasing.

Now, let ξ and η be $\mathcal{S}$- and $\mathcal{T}$-measurable, respectively. We shall require bounds on the covariance of ξ and η. The proofs of the following results follow those given in Billingsley [1].

Lemma 3.1. *Let S, $T \subset \mathbf{Z}^d$ with $\rho(S, T)=k$. If ξ and η are $\mathcal{S}$- and $\mathcal{T}$-measurable. respectively, and if $\xi \in \mathcal{L}^r$, $\eta \in \mathcal{L}^s$, where $1<r<\infty$, and s is the conjugate exponent of r, then*

$$|\mathbf{E}(\xi\eta)-\mathbf{E}(\xi)\mathbf{E}(\eta)| \leq 2\phi^{1/r}(k)\|\xi\|_r\|\eta\|_s. \tag{3.2}$$

Lemma 3.2. *Suppose S, $T \subset \mathbf{Z}^d$ with $\rho(S, T)=k$ assume ξ and η are $\mathcal{S}$- and $\mathcal{T}$-measurable, respectively. Assume, moreover, that ξ, $\eta \in \mathcal{L}^\infty$. Then,*

$$|\mathbf{E}(\xi\eta)-\mathbf{E}(\xi)\mathbf{E}(\eta)| \leq 2\|\xi\|_\infty\|\eta\|_\infty\phi(k). \tag{3.3}$$

It follows from Lemma 3.1, with $r=\frac{1}{2}$, that

$$\sum_{k=1}^{\infty} k^{d-1}\phi^{1/2}(k)<\infty \Rightarrow \sum_x |C(x)| = K_\phi < \infty \tag{3.4}$$

where C is the covariance function of B.

The following result is crucial to the proof of the central limit theorem for $S_n(B)$.

Theorem 3.1. *Suppose B is ϕ-mixing and bounded (i.e., $B(0) \in \mathcal{L}^\infty$). Also assume $EB(0)=0$. Suppose that for each $n \geq 1$, k_n is a nonnegative function on $\mathbf{Z}^d$. Assume the functions k_n are uniformly bounded and that $\sum_x k_n(x)=\mathrm{O}(n)$. Then $\sum k^{d-1}\phi^{1/2}(k)<\infty \Rightarrow$*

$$\sum_{\substack{x_i \in Z^d\\ 1\leq i\leq 4}} \left|\mathbf{E}\left[\prod_{i=1}^{4} B(x_i)h_n(x_i)\right]\right| = \mathrm{O}(n^2). \tag{3.5}$$

Proof. Fix $x_i \in \mathbf{Z}^d$, $1 \leq i \leq 4$, and let $\rho_{IJ}=\min_{i\neq j}\rho_{ij}$ where $\rho_{ij}=\|x_i-x_j\|_\infty$. Select $H \neq I$, J, and let K be such that $\rho_{HK}=\min_{j\neq H}\rho_{Hj}$. Note that $\rho_{HK} \geq \rho_{IJ}$, so that $\phi(\rho_{HK}) \leq \phi(\rho_{IJ})$. Now, by (3.3),

$$\begin{aligned}\left|\mathbf{E}\left[\prod_{i=1}^{4} B(x_i)\right]\right| &\leq 2\|B(0)\|_\infty^2\phi(\rho_{HK})\\ &\leq 2\|B(0)\|_\infty^2\phi^{1/2}(\rho_{HK})\phi^{1/2}(\rho_{IJ}).\end{aligned}$$

Thus,

$$\left|\mathbf{E}\left[\prod_{i=1}^{4} B(x_i)\right]\right| \leq 2\|B(0)\|_\infty^2\left\{\sum_{i\neq j\neq k\neq l} \phi^{1/2}(\rho_{ij})\phi^{1/2}(\rho_{kl}) + \sum_{i\neq j\neq k} \phi^{1/2}(\rho_{ij})\phi^{1/2}(\rho_{jk})\right\}. \tag{3.6}$$

From (3.6) we see that to establish the validity of (3.5) it suffices to show,

$$\sum_{\substack{x_i\in Z^d\\ 1\leq i\leq 4}} \left[\prod_{i=1}^{4} h_n(x_i)\right]\phi^{1/2}(\rho_{12})\phi^{1/2}(\rho_{34}) = \mathrm{O}(n^2) \tag{3.7}$$

and

$$\sum_{\substack{x_i\in Z^d\\ 1\leq i\leq 4}} \left[\prod_{i=1}^{4} h_n(x_i)\right]\phi^{1/2}(\rho_{12})\phi^{1/2}(\rho_{23}) = \mathrm{O}(n^2). \tag{3.8}$$

Since $\sum k^{d-1}\phi^{1/2}(k)<\infty$, we have $\sup_y \sum_x \phi^{1/2}(\rho(x,y))<\infty$, say, equals M. Then, the term on the left of (3.7) is equal to

$$\left[\sum_{x,y} h_n(x)h_n(y)\phi^{1/2}(\rho(x,y))\right]^2 \leq (LM)^2\left[\sum_y h_n(y)\right]^2 = \mathrm{O}(n^2)$$

where $L=\sup_{n,x} h_n(x)$. Moreover, the expression on the left of (3.8) equals

$$\sum_{x_1,x_4} h_n(x_1)h_n(x_4)\left[\sum_{x_2} h_n(x_2)\phi^{1/2}(\rho_{12})\left(\sum_{x_3} h_n(x_3)\phi^{1/2}(\rho_{23})\right)\right]$$
$$\leq (LM)^2\left[\sum_x h_n(x)\right]^2 = \mathrm{O}(n^2).$$

This completes the proof of Theorem 3.1.

4. The asymptotic variance of $S_n(B)$

In this section we shall determine the asymptotic variance of $S_n(B)$. We denote by $N_n(B)$ the occupation time of B by time n of a typical particle. That is, $N_n(B)=\sum_{j=1}^{n} 1_B(\xi_j)$.

Theorem 4.1. *The characteristic function of $S_n(B)$ is given by*

$$\tau_n(\theta)=\mathbf{E}\left\{\exp\sum_x A_0(x)[\ln E_x(\exp(\mathrm{i}\theta N_n(B)))]\right\} \tag{4.1}$$

Proof. Let $\{\xi_{nx}^{(m)}\}_n$ be independent random walks for $m \geqslant 1$ and $x \in \mathbf{Z}^d$, all with transition function P. Then,

$$S_n(B) = \sum_x \sum_{m=1}^{A_0(x)} \sum_{k=1}^{n} 1_B(\xi_{kx}^{(m)})$$

and therefore,

$$\mathbf{E}(\exp[\mathrm{i}\theta S_n(B)] \mid A_0) = \prod_x \{E_x(\exp[\mathrm{i}\theta N_n(B)])\}^{A_0(x)}.$$

Thus, (4.1) follows by taking expectations of this last equation.

Using (4.1) and standard characteristic function arguments we obtain the following facts:

$$\mathbf{E}S_n(B) = n\lambda |B| \tag{4.2}$$

$$\operatorname{Var} S_n(B) = \operatorname{Var}\Big(\sum_x A_0(x) E_x N_n(B)\Big) + \lambda \sum_x \operatorname{Var}_x N_n(B) \tag{4.3}$$

where $\lambda = \mathbf{E}A_0(x)$.

The proof of the next lemma can be found in [7]. Although the results there were proved for $d = 1$, the same arguments work for arbitrary d.

Lemma 4.1. *Suppose $\{\xi_n\}$ is a transient aperiodic random walk. Then*

$$\sum_x E_x N_n(B)^2 \sim n\Big[|B| + 2 \sum_{y \in B} G(y, B)\Big], \quad n \to \infty. \tag{4.4}$$

Lemma 4.2. *Let the hypothesis be as in Lemma* 4.1. *Set $m = \mathbf{E}\xi_1$, if $\xi_1 \in \mathscr{L}^1$, and set $m = \infty$ otherwise. Then,*

$$\sum_x [E_x N_n(B)]^2 = \begin{cases} n|B|^2/|m| + \mathrm{o}(n) & d = 1, \\ \mathrm{o}(n) & d \geqslant 2. \end{cases} \tag{4.5}$$

Proof. The proof of (4.5) for $d = 1$ can be found in [7]. So, assume $d \geqslant 2$. First note that $E_x N_n(B) = G_n(x, B) \leqslant G(x, B)$, and $\sum_x G_n(x, B) = n|B|$. By the renewal theorem (see, e.g. [6]), $G(x, B) \to 0$ as $|x| \to \infty$, and a simple summability argument now shows that $\sum G(x, B) G_n(x, B) = \mathrm{o}(n)$. This completes the proof of the lemma.

From Lemmas 4.1 and 4.2 we obtain the asymptotic behavior of the second term on the right hand side of (4.3):

$$\sum_x \operatorname{Var}_x N_n(B) \sim \begin{cases} n\Big[|B| + 2 \sum_{y \in B} G(y, B) - |B|^2/|m|\Big] & d = 1, \\ n\Big[|B| + 2 \sum_{y \in B} G(y, B)\Big] & d \geqslant 1. \end{cases} \tag{4.6}$$

In handling the first term on the right hand side of (4.3), we impose the condition $\sum k^{d-1}\phi^{1/2}(k)<\infty$, on the mixing function ϕ of the A_0 process.

Lemma 4.3. *Assume* $\sum k^{d-1}\phi^{1/2}(k)<\infty$. *Then,*

$$\operatorname{Var}\Big(\sum_x A_0(x)E_xN_n(B)\Big)=\begin{cases} n\dfrac{|B|^2}{m}\sum C_0(x)+o(n) & d=1,\\ o(n) & d\geqslant 2.\end{cases} \tag{4.7}$$

Proof. By stationarity we have

$$\operatorname{Var}\Big(\sum_x A_0(x)E_xN_n(B)\Big)=\sum_z\Big[\sum_x G_n(x,B)G_n(x+z,B)\Big]C_0(z) \tag{4.8}$$

For dimension $d\geqslant 2$, we have seen that the term in brackets on the right hand side of the above equation is $o(n)$. By (3.4), $\sum|C_0(z)|<\infty$, and so the result for $d\geqslant 2$ follows from the bounded convergence theorem.

For dimension $d=1$, arguments similar to those found in [7] show that

$$\sum_{i,j=1}^{n}\sum_x P_i(x,u)P_j(x+z,v)\sim n|m|^{-1}$$

for fixed u, v, z. Using this fact and bounded convergence we get that the term on the right hand side of (4.8) is asymptotic to $n|B|^2|m|^{-1}\sum C_0(z)$, and this completes the proof of the lemma.

Using (4.6), (4.7), and (4.3), we get the following result.

Theorem 4.2. *Assume A_0 is a strictly stationary, second order process, that is ϕ-mixing with $\sum k^{d-1}\phi^{1/2}(k)<\infty$. Suppose that the random walks by which the particles move are transient and aperiodic. Then*

$$\operatorname{Var} S_n(B)\sim n\sigma^2(B) \tag{4.9}$$

where

$$\sigma^2(B)=\begin{cases}\dfrac{|B|^2}{|m|}\Big[\sum_x C_0(x)-\lambda\Big]+\lambda\Big[|B|+2\sum_{y\in B}G(y,B)\Big] & d=1,\\ \lambda\Big[|B|+2\sum_{y\in B}G(y,B)\Big] & d\geqslant 2.\end{cases} \tag{4.10}$$

5. A central limit theorem for $S_n(B)$

In this section we show that for $d\geqslant 3$, $S_n(B)$ is asymptotically normally distributed. This is first done under the hypothesis that the A_0 process is bounded (i.e.,

$A_0(0)\in\mathscr{L}^\infty$). It is then shown that the boundedness condition can be dropped by making use of a truncation type argument.

In determining the asymptotic behavior of the distribution of $S_n(B)$, the following result will be used, and can be found in [5].

Consider a double sequence of random variables

$$\text{(S)}\qquad \begin{array}{l} X_1^{(1)}, X_2^{(1)}, \ldots, X_{n_1}^{(1)} \\ X_1^{(2)}, X_2^{(2)}, \ldots, X_{n_2}^{(2)} \\ \vdots \quad\ \vdots \qquad\quad \vdots \\ X_1^{(k)}, X_2^{(k)}, \ldots, X_{n_k}^{(k)} \\ \vdots \quad\ \vdots \qquad\quad \vdots \end{array}$$

where the random variables in the same row are defined on the same probability space. For $0\leqslant\alpha\leqslant 1$, let

$$S_\alpha^{(k)}=\sum_{j=1}^{[\alpha n_k]} X_j^{(k)}.$$

Theorem 5.1. *Suppose the r.v.'s in the scheme* (S) *satisfy the following conditions*:

$$\text{(C1)}\qquad \overline{\lim_{k\to\infty}}\,\mathbf{E}\Big(\sum_{j=[\beta n_k]+1}^{[\alpha n_k]} X_j^{(k)}\Big)^2\leqslant\chi(\alpha-\beta),\quad 0\leqslant\beta<\alpha\leqslant 1$$

for some function $\chi(s)$ which is bounded on $[0, 1]$ *and tends to* 0 *as s tends to* 0.

$$\text{(C2)}\qquad \lim_{\Delta\to 0^+}\frac{1}{\Delta}\,\overline{\lim_{k\to\infty}}\,\mathbf{E}\left|\mathbf{E}\Big(\sum_{j=[\alpha n_k]+1}^{[(\alpha+\Delta)n_k]} X_j^{(k)}\Big|S_\alpha^{(k)}\Big)-\Delta\rho(\alpha)S_\alpha^{(k)}\right|=0$$

for $0\leqslant\alpha<1$, where ρ is some continuous function on $[0, 1)$.

$$\text{(C3)}\qquad \lim_{\Delta\to 0^+}\frac{1}{\Delta}\,\overline{\lim_{k\to\infty}}\,\mathbf{E}\left|\mathbf{E}\Big(\Big[\sum_{j=[\alpha n_k]+1}^{[(\alpha+\Delta)n_k]} X_j^{(k)}\Big]^2\Big|S_\alpha^{(k)}\Big)-\Delta\sigma^2(\alpha)\right|=0$$

for $0\leqslant\alpha<1$, where σ^2 is some continuous function on $[0, 1)$.

$$\text{(C4)}\qquad \lim_{\Delta\to 0^+}\frac{1}{\Delta}\,\overline{\lim_{k\to\infty}}\int_{|x|>\varepsilon} x^2\,\mathrm{d}G_{\alpha,\Delta}^{(k)}(x)=0,\quad 0\leqslant\alpha<1$$

for every $\varepsilon>0$, where $G_{\alpha,\Delta}^{(k)}$ is the distribution function of $\sum_{j=[\alpha n_k]+1}^{[(\alpha+\Delta)n_k]} X_j^{(k)}$.

Then, for $0\leqslant\alpha\leqslant 1$, $S_\alpha^{(k)}\xrightarrow{\mathscr{D}}\mathcal{N}(0,\psi(\alpha))$, where

$$\psi(\alpha)=\begin{cases}\displaystyle\int_0^\alpha \sigma^2(s)\exp\Big(2\int_s^\alpha \rho(u)\,\mathrm{d}u\Big)\,\mathrm{d}s, & 0\leqslant\alpha<1,\\[2ex] \displaystyle\lim_{s\to 1}\psi(s), & \alpha=1.\end{cases}$$

We shall apply this theorem to our problem with $\rho = 0$, $\sigma^2 = 1$, $n_k = k$, and

$$X_j^{(k)} = [A_j(B) - \lambda|B|]/[\mathrm{Var}\, S_k(B)]^{1/2}. \tag{5.1}$$

In verifying (C1)–(C4) we assume the A_0 process is strictly stationary and ϕ-mixing with $\sum k^{d-1}\phi^{1/2}(k) < \infty$. In establishing (C4) we shall in addition assume the A_0 process is bounded. But, the boundedness condition will be subsequently removed.

Throughout the remainder of the paper we shall use the following convention. When we put a noninteger in a position where there should obviously be an integer, then it is to be interpreted as its integral part.

The proofs of the following results are analogous to those that can be found in [2]. In Lemma 5.2 we use the fact that, $\sup_x P_n(0, x) = O(n^{-d/2})$. This holds for any strongly aperiodic random walk, in view of Theorem 7.1, Proposition 7.6, and Proposition 7.8 of [6].

Lemma 5.1. *For $x, y \in \mathbf{Z}^d$, and $0 \leq i < j$,*

$$\mathrm{Cov}(A_i(x), A_j(y)) = \sum_z C_i(x - z) P_{j-1}(z, y). \tag{5.2}$$

Lemma 5.2. *Assume $\sum k^{d-1}\phi^{1/2}(k) < \infty$. Then there is a constant A (depending only on the random walk), such that for $x \neq 0$,*

$$|C_n(x)| \leq A n^{-d/2}\Big(\sum_z |C_0(z)| + \lambda\Big). \tag{5.3}$$

Lemma 5.3. *Let the hypothesis be as in Lemma 5.2. Then,*

$$\sum_x |C_n(x)| = O(1). \tag{5.4}$$

Lemma 5.4. *Let the hypothesis be as in Lemma 5.2. Then,*

$$\mathrm{Var}\, A_n(B) = O(1). \tag{5.5}$$

We now show that the random variables in (5.1) satisfy condition (C1) of Theorem 5.1, with $\chi(s) = Ms$, for some constant M.

Theorem 5.2. *Assume $\sum k^{d-1}\phi^{1/2}(k) < \infty$, and let $X_j^{(k)}$ be defined as in (5.1). Then there is a constant M such that*

$$\overline{\lim_{k\to\infty}}\, \mathbf{E}\Big(\sum_{j=\beta k+1}^{\alpha k} X_j^{(k)}\Big)^2 \leq M(\alpha - \beta), \quad 0 \leq \beta < \alpha \leq 1. \tag{5.6}$$

Proof. Using Theorem 2.1, we find that

$$\mathbf{E}\Big(\sum_{j=\beta k+1}^{\alpha k} X_j^{(k)}\Big)^2 = \sigma_k^{-1}\Big[\sum_{\beta k+1}^{\alpha k} \mathrm{Var}\, A_j(B) + 2\sum_{i<j} \mathrm{cov}(A_i(B), A_j(B))\Big]$$

where $\sigma_k = \operatorname{Var} S_k(B)$. Lemma 5.4 implies that the first sum inside the brackets is $O((\alpha-\beta)k)$. Using Lemmas 5.1 and 5.3 we find that

$$\begin{aligned}
&\Big|\sum_{i<j} \operatorname{cov}(A_i(B), A_j(B))\Big| \\
&\quad\leqslant \sum_{i=\beta k+1}^{\alpha k-1} \sum_{x,y\in B} \sum_{z} |C_i(z)| G_{\alpha k-i}(x, y+Z) \\
&\quad\leqslant |B|^2 \sup_x G(0,x) \sum_{i=\beta k+1}^{\alpha k-1} \sum_z |C_i(z)| = O((\alpha-\beta)k).
\end{aligned}$$

By Theorem 4.1, $\sigma_k \sim k\sigma^2(B)$, and the result follows.

Before we show that the random variables in (5.1) satisfy conditions (C2) and (C3) we need some preliminary results. First some notation: For $k \in \mathcal{N}$ and $0 \leqslant \alpha \leqslant 1$, let $\xi_{j,\alpha k} = \xi_{\alpha k+j}$, $j = 0, 1, 2, \ldots$. Then,

$$H_{\alpha,\Delta k}(B) = \sum_{j=1}^{\Delta k} 1_B(\xi_{j,\alpha k}) \tag{5.7}$$

represents the occupation time of B be a typical particle from time $\alpha k+1$ to time $(\alpha+\Delta)k$.

The proofs of the following lemmas are similar to those used in establishing Theorem 4.1 and (4.6).

Lemma 5.5. *Let* $T_{\alpha,\Delta k}(B) = \sum_{j=\alpha k+1}^{(\alpha+\Delta)k} (A_j(B) - \lambda|B|)$. *Then,*

$$\begin{aligned}
&\mathbf{E}(\exp \mathrm{i}\theta T_{\alpha,\Delta k}(B) \mid A_{\alpha k}) \\
&\quad= \exp\Big(\Big[\sum_x A_{\alpha k}(x) \ln(E_x(\exp \mathrm{i}\theta H_{\alpha,\Delta k}(B)))\Big] - \mathrm{i}\theta\lambda|B|\Delta k\Big).
\end{aligned} \tag{5.8}$$

Lemma 5.6. *For dimension* $d \geqslant 2$,

$$\sum_x \operatorname{Var}_x H_{\alpha,\Delta k}(B) \sim k\Delta\gamma^2(B), \tag{5.9}$$

where $\gamma^2(B) = |B| + 2\sum_{y\in B} G(y, B)$.

Using Lemma 5.5 we get the following results. With probability one,

$$\mathbf{E}\Big(\sum_{\alpha k+1}^{(\alpha+\Delta)k} (A_j(B) - \lambda|B|)\Big| A_{\alpha k}\Big) = \sum_x (A_{\alpha k}(x) - \lambda|B|) G_{\Delta k}(x, B) \tag{5.10}$$

$$\begin{aligned}
&\mathbf{E}\Big(\Big(\sum_{\alpha k+1}^{(\alpha+\Delta)k} (A_j(B) - \lambda|B|)\Big)^2 \Big| A_{\alpha k}\Big) \\
&\quad= \Big[\sum_x (A_{\alpha k}(B) - \lambda|B|) G_{\Delta k}(x, B)\Big]^2 + \sum_x A_{\alpha k}(x) \operatorname{Var}_x H_{\alpha,\Delta k}(x, B).
\end{aligned} \tag{5.11}$$

The following lemma will also be used in the sequel.

Lemma 5.7. *Let X be a random variable on $(\Omega, \mathcal{A}, P)$. Assume $\mathcal{G}_1$ and $\mathcal{G}_2$ are sub σ-fields of $\mathcal{A}$ with $\mathcal{G}_1 \subset \mathcal{G}_2$. Assume ϕ is convex on $\mathcal{R}$, and that X and $\phi(E(X \mid \mathcal{G}_2))$ are integrable. Then*

$$E[\phi(E(X \mid \mathcal{G}_1))] \leq E[\phi(E(X \mid \mathcal{G}_2))] \tag{5.12}$$

Proof. Using Jensen's inequality, and standard results on conditional expectation we obtain, that with probability one

$$\phi(E(X \mid \mathcal{G}_1)) = \phi(E(E(X \mid \mathcal{G}_2) \mid \mathcal{G}_1)) \leq E(\phi(E(X \mid \mathcal{G}_2)) \mid \mathcal{G}_1)$$

and the result follows by taking expectations.

We now show that the random variables in (5.1) satisfy conditions (C2) and (C3) with $\rho(\alpha) \equiv 0$ and $\sigma^2(\alpha) \equiv 1$.

Theorem 5.3. *For dimension $d \geq 3$ we have*

$$\lim_{k\to\infty} \mathbf{E}\left|\mathbf{E}\left(\sum_{j=\alpha k+1}^{(\alpha+\Delta)k} X_j^{(k)} \middle| S_\alpha^{(k)}\right)\right| = 0, \quad 0 \leq \alpha \leq 1 \tag{5.13}$$

for each $\Delta > 0$, where $X_j^{(k)}$ is given by (5.1).

Proof. For convenience, set $\sigma_k = [\operatorname{Var} S_k(B)]^{1/2}$. Using (5.10) and Lemma 5.7, we get

$$\begin{aligned}\sigma_k \mathbf{E}\left|\mathbf{E}\left(\sum X_j^{(k)} \mid S_\alpha^{(k)}\right)\right| &\leq \mathbf{E}\left|\mathbf{E}\left(\sum (A_j(B) - \lambda|B|)\right) \mid A_{\alpha k}\right)| \\ &\leq \sum_{u \in B} \left(\sum_x \operatorname{Var} A_{\alpha k}(x)[G_{\Delta k}(x, u)]^2 \right. \\ &\qquad \left. + \sum_{x \neq y} C_{\alpha k}(x-y) G_{\Delta k}(x, u) G_{\Delta k}(y, u)\right)^{1/2}.\end{aligned}$$

Using Lemmas 4.2 and 5.4, we conclude that

$$\sum \operatorname{Var} A_{\alpha k}(x)[G_{\Delta k}(x, u)]^2 = o(k) \quad \text{for } d \geq 3.$$

Moreover, using Lemma 5.2, and the fact that $\sum G_n(x, y) = n$, we get that for $d \geq 3$,

$$\sum_{x \neq y} C_{\alpha k}(x-y) G_{\Delta k}(x, u) G_{\Delta k}(y, u) = o(k).$$

The result now follows from Theorem 4.2.

Theorem 5.4. *For dimension $d \geq 3$,*

$$\lim_{k\to\infty} \mathbf{E}\left|\mathbf{E}\left(\left(\sum_{j=\alpha k+1}^{(\alpha+\Delta)k} X_j^{(k)}\right)^2 \middle| S_\alpha^{(k)}\right) - \Delta\right| = 0 \tag{5.14}$$

for $0 \leq \alpha \leq 1$, for each $\Delta > 0$, where $X_j^{(k)}$ is given by (5.1).

Proof. Again, set $\sigma_k = [\operatorname{Var} S_k(B)]^{1/2}$. Using Lemma 5.7 and (5.11), we get that

$$\sigma_k^2 \mathbf{E}|\mathbf{E}((\sum X_j^{(k)})^2 | S_\alpha^{(k)}) - \Delta|$$
$$\leq \mathbf{E}\left|\mathbf{E}\left(\left(\sum_{\alpha k+1}^{(\alpha+\Delta)k} (A_j(B) - \lambda|B|)\right)^2 \middle| A_{\alpha k}\right) - \Delta\sigma_k^2\right|$$
$$\leq \mathbf{E}\left(\sum_x (A_{\alpha k}(B) - \lambda|B|) G_{\Delta k}(x, B)\right)^2 + \mathbf{E}\left|\sum_x A_{\alpha k}(x) \operatorname{Var}_x H_{\alpha,\Delta k}(B) - \Delta\sigma_k^2\right|.$$

Just as in the proof of Theorem 5.3 we can show that the first term on the right hand side of the last inequality is $o(k)$, for $d \geq 3$.

To handle the second term on the right hand side of the last inequality, first note that Theorem 4.2 and Lemma 5.6 imply that

$$\lambda \sum_x \operatorname{Var}_x H_{\alpha,\Delta k}(B) - \Delta\sigma_k^2 = o(k) \tag{5.15}$$

for $d \geq 2$. Now, it is easy to see that $\sup_x \operatorname{Var}_x H_{\alpha,\Delta k}(B) = O(1)$. Using this along with Lemmas 5.2, 5.4, and 5.6, we get, by the Cauchy–Schwarz inequality that

$$\mathbf{E}\left|\sum_x A_{\alpha k}(x) \operatorname{Var}_x H_{\alpha,\Delta k}(B) - \lambda \sum_x \operatorname{Var}_x H_{\alpha,\Delta k}(B)\right| = o(k) \tag{5.16}$$

for $d \geq 3$. The result in (5.14) now follows in view of (5.15), (5.16), and Theorem 4.2.

It remains to verify (C4) for the random variables in (5.1). To do this it is clearly sufficient to prove that for some $\delta > 0$,

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta} \overline{\lim_{k\to\infty}} \mathbf{E}\left|\sum_{\alpha k+1}^{(\alpha+\Delta)k} X_j^{(k)}\right|^{2+\delta} = 0, \quad 0 \leq \alpha \leq 1. \tag{5.17}$$

We shall verify (5.17) with $\delta = 2$.

Theorem 5.5. *Let the random variables $X_j^{(k)}$ be defined as in* (5.1). *Assume that A_0 is bounded. Then,*

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta} \overline{\lim_{k\to\infty}} \mathbf{E}\left[\sum_{\alpha k+1}^{(\alpha+\Delta)k} X_j^{(k)}\right]^4 = 0, \quad 0 \leq \alpha \leq 1. \tag{5.18}$$

Proof. The details of the proof of this theorem are extremely long and messy. Therefore, we shall only give an outline of the proof. The idea of the proof is to show that

$$\mathbf{E}\left(\sum_{j=\alpha k+1}^{(\alpha+\Delta)k} (A_j(B) - \lambda|B|)\right)^4 = O((\Delta k)^2). \tag{5.19}$$

From this, and Theorem 4.2 it will then follow that

$$\overline{\lim_{k\to\infty}}\, \mathbf{E}\left[\sum_{\alpha k+1}^{(\alpha+\Delta)k} X_j^{(k)}\right]^4 \leqslant O(\Delta^2),$$

and so (5.18) will be established.

The fourth moment in (5.19) can be computed by using characteristic function arguments, along with Lemma 5.5. The resulting expression involves basically five types of different terms. We then show that each of these terms is $O((\Delta k)^2)$. The following facts give this result for all but the last of the five terms.

$$\sum_x E_x H_{\alpha,\Delta k}(B)^j = O(\Delta k), \quad 1 \leqslant j \leqslant 4,$$

$$\sum_x [E_x H_{\alpha,\Delta k}(B)]^2 = O(\Delta k),$$

$$\sup_{x,k} E_x H_{\alpha,\Delta k}(B)^j = O(1), \quad 1 \leqslant j \leqslant 2,$$

and

$$\sum k^{d-1}\phi^{1/2}(k) < \infty \Rightarrow \sum |C_0(x)| < \infty.$$

The last of the five terms in the expression for the fourth moment in (5.19) is

$$\mathbf{E}\left[\sum_x (A_0(x) - \lambda) E_x H_{\alpha,\Delta k}(B)\right]^4. \tag{5.20}$$

If we let $B(x) = A_0(x) - \lambda$, and $h_n(x) = E_x H_{\alpha,n}(B)$, then Theorem 3.1 shows that the term in (5.20) is $O((\Delta k)^2)$.

Letting $\alpha = 1$ in Theorem 5.1, we now have the following limit theorem.

Theorem 5.6. *Suppose A_0 is strictly stationary, bounded, and ϕ-mixing with $\sum k^{d-1}\phi^{1/2}(k) < \infty$. Also assume that the random walks by which the particles move are strongly aperiodic. Then for $d \geqslant 3$,*

$$\frac{S_n(B) - ES_n(B)}{\sqrt{\operatorname{Var} S_n(B)}} \xrightarrow{\mathscr{D}} \Phi \tag{5.21}$$

where Φ is the standard normal distribution.

Finally, we shall show that the boundedness condition on the A_0 process can be omitted. The following preliminary result is needed.

Lemma 5.8. *Suppose A is ϕ-mixing, and let f be a real-valued Borel measurable function. Then the process $f \circ A$ is ϕ-mixing with the same ϕ function as A. Also $f \circ A$ is strictly stationary if A is.*

The proof of the lemma follows directly from the fact that if $S \subset \mathbf{Z}^d$, and $\mathcal{S}$, $\hat{\mathcal{S}}$ are the σ-fields generated by $A(x)$, $x \in S$ and $f \circ A(x)$, $x \in S$, respectively, then $\hat{\mathcal{S}} \subset \mathcal{S}$.

We now show that the boundedness condition on the A_0 process in Theorem 5.6 can be removed.

Theorem 5.7. *Suppose A_0 is a second order, strictly stationary process, which is ϕ-mixing with $\sum k^{d-1}\phi^{1/2}(k)<\infty$. Also assume that the random walks by which the particles move are strongly aperiodic. Then, for $d \geq 3$, (5.21) holds.*

Proof. For fixed positive u, define $f_u(t)=t\, 1_{[-u\,u]}(t)$, and set $g_u(t)=t-f_u(t)$. By Lemma 5.8, the processes $f_u \circ A_0$ and $g_u \circ A_0$ are ϕ-mixing. Moreover, $f_u \circ A_0$ is bounded. Let $T_{n,u}(B)$ and $U_{n,u}(B)$ be the total occupation time of B by time n of the particles induced by the processes $f_u \circ A_0$ and $g_u \circ A_0$, respectively. From Theorem 5.6 we have

$$\frac{T_{n,u}(B)-n|B|\alpha(u)}{\sqrt{\operatorname{Var} T_{n,u}(B)}} \xrightarrow{\mathscr{D}} \Phi \tag{5.22}$$

where $\alpha(u)=E[f_u \circ A(0)]$.

Now set $\beta(u)=E[g_u \circ A(0)]$. Arguments as in Theorem 4.2 show that $\operatorname{Var} T_{n,u}(B) \sim n\gamma(B)\alpha(u)$, and $\operatorname{Var} U_{n,u}(B) \sim n\gamma(B)\beta(u)$, where $\gamma(B)=|B|+2\sum_{y\in B} G(y,B)$. Set,

$$Z_n(B)=[S_n(B)-n\lambda|B|]/[n\lambda\gamma(B)]^{1/2},$$

$$X_{n,u}(B)=[T_{n,u}(B)-n|B|\alpha(u)]/[n\lambda\gamma(B)]^{1/2},$$

$$Y_{n,u}(B)=[U_{n,u}(B)-n|B|\beta(u)]/[n\lambda\gamma(B)]^{1/2}.$$

Then, for each u, $Z_n(B)=X_{n,u}(B)+Y_{n,u}(B)$.

Let F be a closed subset of $\mathscr{R}$, and for $\varepsilon>0$, set $F_\varepsilon=\{x:\rho(F,x)\leq\varepsilon\}$. Then,

$$\mathbf{P}(Z_n(B)\in F)\leq \mathbf{P}(|Z_n(B)-X_{n,u}(B)|\geq\varepsilon)+\mathbf{P}(X_{n,u}(B)\in F_\varepsilon). \tag{5.23}$$

Now, it follows from (5.22) that

$$X_{n,u}(B)\xrightarrow{\mathscr{D}}(\alpha(u)/\lambda)^{1/2}X \tag{5.24}$$

where X is standard normal. Setting $X(u)=(\alpha(u)/\lambda)^{1/2}X$, we have that

$$\varlimsup_{n\to\infty}\mathbf{P}(X_{n,u}(B)\in F_\varepsilon)\leq \mathbf{P}(X(u)\in F_\varepsilon). \tag{5.25}$$

By Chebychev's inequality, and our above results on the asymptotic behavior of $\operatorname{Var} U_{n,u}(B)$, we can conclude that

$$\varlimsup_{n\to\infty}\mathbf{P}(|Z_n(B)-X_{n,u}(B)|\geq\varepsilon)\leq\beta(u)/\lambda\varepsilon^2. \tag{5.26}$$

Since $A_0(0)$ has finite mean, $\beta(u) \to 0$ and $\alpha(u) \to \lambda$, as $u \to \infty$. Using this, the fact that F_ε is closed, and (5.23)–(5.26) we conclude that for $\varepsilon > 0$,

$$\overline{\lim_{n \to \infty}} \mathbf{P}(Z_n(B) \in F) \leq \mathbf{P}(X \in F_\varepsilon).$$

Letting $\varepsilon \downarrow 0$, we get that $Z_n(B) \xrightarrow{\mathscr{D}} X$ as desired.

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